A MIXED PIVOTING APPROACH TO THE FACTORIZATION OF INDEFINITE MATRICES IN POWER SYSTEM STATE ESTIMATION

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Abstract – This paper presents a robust and efficient sparsity oriented approach to the factorization of indefinite matrices that appear in connection with a variety of power system state estimation formulations. Particularly, the problem of representing zero injections as equality constraints in the normal equations formulation is fully discussed and analyzed. A new ordering scheme for mixed 1×1, 2×2 pivoting is proposed and tested. Tests on two real life systems (41 buses and 1331 buses) are reported and discussed.

Keywords: State estimation; indefinite matrix; equality constraints; tableau formulation; mixed 1×1, 2×2 pivoting.

1. INTRODUCTION

The state estimation model [1,2] is given by

$$z = h(x) + w \tag{1}$$

where z is the known (m×1) vector of measurements, h(.) is a (m×1) vector of non–linear functions, x is the unknown (2n×1) vector of state variables, w represents the noise associated to the raw measurements, m is the number of measurements, and n is the number of buses.

A popular approach consists in minimizing the performance index

$$J(x) \triangleq r'r \tag{2}$$

$$r \triangleq z-h(x) \tag{3}$$

where r' is the transpose of r. (Indeed, we should have written r'Wr, where W is the (m×m) diagonal matrix of weighting factors, but for notational simplicity a transformed measurement vector given by $W^{1/2}.z$ is used instead; notice that h(x) and r are modified accordingly, and so will be the Jacobian matrix $\partial h/\partial x$).

The unconstrained minimization of (2) requires that the gradient $\partial J/\partial x$ be zero at the solution point, which leads to the non–linear equation,

$$H'(\hat{x}).(z - h(\hat{x})) = 0 \tag{4}$$

where $\hat{x}$ is the state estimate and H is the Jacobian matrix

$$H \triangleq \frac{\partial h}{\partial x} \tag{5}$$

Equation (4) can be solved by an iterative method which computes the correction Δx^k at each iteration by the system of linear equations

$$H'(x^k).H(x^k).\Delta x^k = H'(x^k).\Delta z(x^k) \tag{6}$$

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$$\Delta z(x^k) = z - h(x^k) \tag{7}$$

$$x^{k+1} = x^k + \Delta x^k \tag{8}$$

for k = 0,1,2... until appropriate convergence is attained.

An important feature of the normal equations approach, eq. (6), is that the gain matrix H'H is positive definite, provided that the system is observable. And so, sparse triangular factorization does not require pivoting for numerical stability. Whatever the pivot ordering, a zero pivot will mean system unobservability.

Though formulation (6)–(7) is attractive for its simplicity, there are practical situations in which the normal equations approach is not robust enough to yield a converged solution. Among the main causes of numerical problems we can point out the presence of injection measurements [9], the use of high weighting factors to enforce zero injection pseudo–measurements [5], and the presence of adjacent branches with a wide range of series susceptance (say, long and very short lines, with flow measurements, connected to the same node) [8].

Several alternatives have been proposed to cope with such situations and to improve robustness: handling of zero injection as equality constraints [5], orthogonal transformation method [6–8], method of Peters and Wilkinson [9], and the Hachtel's method [10] (a comprehensive review is found in [2], and results of extensive comparative testing are reported in [3]). More recently, the blocked sparse matrix formulation [13,14], inspired by the Newton optimal power flow approach of [15], has been proposed.

This paper explores a family of methods related to the tableau formulation [10–12], which has been originally suggested for finite element assembly [10], and subsequently has been successfuly applied to circuit analysis [11] and power system state estimation [12]. The main motivation for the tableau formulation is to deal with the state estimation equations in an unsquared form [13], as opposed to the usual squared form H'H associated to the normal equations approach and to some related methods. The tableau formulation brings potential benefits both to the numerical conditioning of the problem, as well as to the sparsity of the triangular factors; so in principle we would have improved robustness along with computational efficiency. The major dificulty with the method is the same problem that has plagued the equality constraint method [5], that is, we have to factorize an indefinite matrix, as opposed to the positive definite matrix of the normal equations approach. The factorization of an indefinite matrix may lead to zero pivots even when the system is observable (and so the matrix is non–singular). It has been recognized that the zero–pivot postponement technique may introduce numerical instabilities, even though it has been successfully used in a number of cases. On the other hand, the general purpose Harwell routines adopted in [12] are based on a robust hybrid 1×1 and 2×2 pivot strategy in which the factorization method takes into account both the sparsity of the resulting triangular factors as well as the numerical values of the pivots. It has been shown [16] that, if the matrix is non–singular, when zero pivots (zero diagonal elements) avoid the progress of the factorization process, it is always possible to find a non–singular 2×2 pivot such that the pivoting will be possible (typically, a 2×2 pivot will have one zero diagonal element and non–zero off–diagonal elements).

This paper proposes an efficient mixed 1×1, 2×2 pivot scheme for indefinite matrices originated in a variety of tableau formulations. Contrary to what one could expect from common sense analysis the method is extremely simple. Our present implementation has been derived from a conventional package originally developed for positive definite matrices (normal equations). Only minor modifications were required. Though the proposed method follows closely the one given in [12], it uses especially designed routines that profit from the specific features of the state estimation problem.

2. SPARSE TABLEAU FORMULATION

In this section we derive and discuss the main features of the unsquared form of the state estimation equations.

Motivation

After going through the k–th iteration the updated residual will be

$$r^{k+1} = z - h(x^{k+1}) \tag{9}$$

Let γ^{k+1} be the linear approximation to r^{k+1} given by

$$\gamma^{k+1} = \Delta z(x^k) - H(x^k).\Delta x^k \tag{10}$$

It can be easily verified that

$$H'(x^k).\gamma^{k+1} = 0 \tag{11}$$

that is to say, if the model is linear the solution point is reached in one iteration. Now let us put eqs. (10) and (11) in the sparse tableau form:

$$\begin{bmatrix} I & H \\ H' & 0 \end{bmatrix} \begin{bmatrix} \gamma^{k+1} \\ \Delta x^k \end{bmatrix} = \begin{bmatrix} \Delta z(x^k) \\ 0 \end{bmatrix} \tag{12}$$

where I is the unit matrix (an alternative derivation of eq. (12) is given in [2]: the objective function (2) is minimized subjected to the constraint (3); γ are the Lagrange multipliers). If we factorize (12) pivoting first on unit matrix I, we bring the system back to the squared form (normal equations):

$$\begin{bmatrix} I & H \\ 0 & -H'H \end{bmatrix} \begin{bmatrix} \gamma^{k+1} \\ \Delta x^k \end{bmatrix} = \begin{bmatrix} \Delta z(x^k) \\ -H'\Delta z(x^k) \end{bmatrix} \tag{13}$$

Though systems (12) and (13) are mathematically equivalent and produce the same solution for infinite precision, in practice the solutions of the two systems of equations may lead to completely different results, as is shown in the following example:

Example 1

We assume a linear DC power flow model for the 3–bus system of Fig. 1:

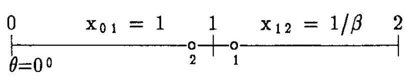

Fig.1: 3–bus example system [8].

The Jacobian and the gain matrix are given by

$$H = \begin{bmatrix} 1 & 0 \\ \beta & -\beta \end{bmatrix} \qquad H'H = \begin{bmatrix} 1+\beta^2 & -\beta^2 \\ -\beta^2 & \beta^2 \end{bmatrix}$$

If, for instance, we consider $\beta = 10^4$, and if the machine we are using is accurate to the 7–th digit, then matrix H'H will become singular

$$H'H = \begin{bmatrix} \beta^2 & -\beta^2 \\ -\beta^2 & \beta^2 \end{bmatrix}$$

which means numerical unobservability. Or, in a less dramatic situation, the matrix may be nearly singular, which may cause numerical instability. On the other hand, the tableau in eq. (13) can be rearranged as

$$\begin{bmatrix} 1 & & 1 & \\ & 1 & \beta & -\beta \\ 1 & \beta & 0 & \\ & -\beta & & 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} 1 & 1 & & \\ 1 & & \beta & \\ & \beta & 1 & -\beta \\ & & -\beta & \end{bmatrix}$$

whose factorization, under the same assumptions, leads to:

$$\begin{bmatrix} 1 & 1 & & \\ & -1 & \beta & \\ & & \beta^2 & -\beta \\ & & & -1 \end{bmatrix}$$

And an accurate solution for the proposed problem can then be obtained from the triangularized matrix. The same example can also be used to illustrate how the conditioning of the gain matrix varies as the distance from singularity decreases due to the increase in the ratio between the susceptances of the two branches ("long" and "short" lines). It is noteworthy that most of the known state estimation methods (i.e. methods that rely on squared flow measurement Jacobians, H'H) will potentially suffer from this type of ill–conditioning. In the following we will use a slightly modified version of the same example to illustrate how the high weighting factors normally used to represent zero–injections may cause the same kind of damage on problem conditioning.

Equality Constraints

Let us consider the minimization problem

$$\text{Min} \quad J(x) = 1/2\, r'r$$
$$\text{s.t} \quad r = z - h(x) \tag{14}$$

The associated Lagrangean function is given by

$$L(x,\gamma) = J(x) - \gamma'(r - z + h(x)) \tag{15}$$

It can be easily shown that the necessary conditions for optimality ($\partial L/\partial x = 0$ and $\partial L/\partial \gamma = 0$) lead to the sparse tableau formulation (12).

In the previous formulation, eq. (14), zero injections are treated as soft constraints, which means that they are not precisely enforced, in principle, to zero the residuals associated to a zero–injection pseudo–measurement. To achieve this objective it would be necessary to use extremely high weighting factors (except, of course, in those cases in which the zero–injection measurements are critical for system observability [4]). To avoid potential numerical problems caused by high weightings the treatment of zero–injections as equality constraints has been suggested [5]. Mathematically this means that problem (14) has to be reformulated to explicity represent the zero–injections as equality–constraints (hard–constraints). The modified problem is

$$\begin{array}{lll} \text{Min} & J(x) = 1/2\, r'r & \\ \text{s.t.} & r = z - h(x) & \\ & g(x) = 0 & \end{array} \qquad (16)$$

where the non–linear set of extra equations $g(x)=0$ represents the zero–injections constraints. Accordingly the Lagrangean function becomes,

$$L(x,\gamma,\lambda) = J(x) - \gamma'(r-z+h(x)) - \lambda'\, g(x) \qquad (17)$$

Minimum point necessary conditions ($\partial L/\partial x=0$, $\partial L/\partial\gamma=0$ and $\partial L/\partial\lambda=0$) can be summarized by the augmented tableau (Hachtel's approach [2,12]):

$$\begin{bmatrix} I & 0 & H \\ 0 & 0 & G \\ H' & G' & 0 \end{bmatrix} \begin{bmatrix} \gamma^{k+1} \\ \lambda^{k+1} \\ \Delta x^k \end{bmatrix} = \begin{bmatrix} \Delta z(x^k) \\ -g(x^k) \\ 0 \end{bmatrix} \qquad (18)$$

The Gaussian elimination of variables γ leads to the form originally suggested in [5], in which regular measurements appear in squared form (submatrix H'H):

$$\begin{bmatrix} 0 & G \\ G' & H'H \end{bmatrix} \begin{bmatrix} \lambda^{k+1} \\ \Delta x^k \end{bmatrix} = \begin{bmatrix} -g(x^k) \\ -H'\Delta z \end{bmatrix} \qquad (19)$$

Example 2

Now we consider a modified version of the 3–bus system of example–1 to show how the conditioning of the gain matrices that appear in eqs. (13) and (19) may be affected by the way zero–injections are represented (measurement or constraint). The modified 3–bus system is given in Fig. 2.

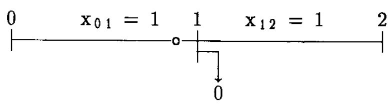

Fig.2: Modified 3–bus example system

Let us first consider the case in which the zero–injection is represented as a pseudo–measurement with weighting w. Jacobian and gain matrices will be

$$H = \begin{bmatrix} 1 & 0 \\ w^{1/2} & -w^{1/2} \end{bmatrix} \qquad H'H = \begin{bmatrix} 1+w & -w \\ -w & w \end{bmatrix}$$

(The weighting associated to the flow measurement is considered to be equal to unit). The bigger the magnitude of w, the closer gain matrix H'H will get to singularity; say for seven–figure floating point arithmetic, $w=10^8$ will make the matrix singular. This is in contrast to what is obtained from formulation (19). In this case tableau (19) assumes the form:

	ec	1	2
ec		2	-1
1	2	1	
2	-1		

which does not depend on w, and can be factorized if we use proper recordering,

	1	ec	2
1	1	2	
ec	2		-1
2		-1	

$\Longrightarrow$

1	2	
	-4	1/4
		-1/4

Indefinite Matrices

From the previous discussion on sparse tableau formulations we can conclude that improved robustness may be achieved as compared to some related methods in which the Jacobian matrix, or a part of it, appears in the squared form H'H. The main difficulty is the need to factorize indefinite matrices (such as the ones in eqs. (12), (18) and (19)). Ref. [5] suggests the delayed elimination scheme in which a zero, or not big enough pivot, is delayed to a latter stage of the factorization process. Other alternatives that appear in the mathematical literature such as partial pivoting have never been seriously considered for they destroy matrix symmetry. Ref. [12] uses Harwell's routines based on mixed 1×1, 2×2 pivoting, which is a sound and proved mathematical technique applicable to a variety of indefinite matrix related problems. Both the delayed elimination scheme, the artial pivoting method, and the mixed 1×1, 2×2 pivoting strategy perform factorization taking into account not only sparsity but pivot magnitudes as well (and so some testing on pivot is carried out during factorization). Refs. [13, 14], on the other hand, try to put matrix elements into blocks in such a way that factorization may proceed as in the positive definite case, that is, symbolic factorization is carried out without bothering about pivot magnitudes. Some care has to be taken, however, to make sure no zero pivots will pop up during numerical factorization [13].

In the next section we present a new ordering scheme suited for mixed 1×1, 2×2 pivoting. The main advantage of the method is that it allows the development of efficient factorization routines that can profit from the special characteristics of the problem. The initial version of the method has been developed to deal with equality constrained normal equations, but it is extendable to other tableau formulations discussed in this section.

3. FACTORIZATION SCHEME

In this section we discuss the basic features of the proposed factorization scheme. Mixed 1×1 and 2×2 pivoting is used. Though the method is intended to be applied to indefinite matrices, it can also be used to factorize positive definite matrices (normal equations), in which case the elimination process will follow the ordering given by the Tinney II scheme, as usual. In fact this is crucial to the success of the method: at each step of the factorization process a new candidate pivot row is selected based on the Tinney II scheme. Only if the corresponding diagonal element is zero (or deemed too small regarding numerical stability), one switches to 2×2 pivoting; a generalized version of the Tinney II is used to select the companion row in the 2×2 pivoting.

Basic Algorithm

For $\gamma = 1,n$

i) Select the next pivoting row based on Tinney II ordering scheme (the k–th row is selected; d_k is the corresponding diagonal element)

ii) If $|d_k| \leq \delta$
then
a) carry out 1×1 pivoting as usual
b) $\gamma \leftarrow \gamma + 1$
else
a) select the companion row based on the extended Tinney II criterion (the i–th row is selected; d_i is the corresponding diagonal element)

b) perform 2×2 pivoting to zero the elements of the k–th and i–th columns
c) $\gamma \leftarrow \gamma + 2$

Remarks:

- For most of the cases 1×1 pivoting will take place. For instance, a 1331 bus system with 417 equality constraints, lead to a gain matrix of order 1748 (eq. (19), fast decoupled state estimator); considering $\delta = 0$, factorization is performed with 1494 1×1 pivots and 127 2×2 pivots. Increasing tolerance to $\delta = 10^{-4}$, causes the number of 2×2 pivots goes to 165 (1418 1×1 pivots).

- Most of the 2×2 pivots will have $d_k = 0$, i.e. they will be either

0	x
x	d_i

or

0	x
x	0

when $d_k \neq 0$ the 2×2 pivots will be either

d_k	x
x	0

or

d_k	x
x	d_i

(cases in which both d_k and d_i are nonzero comparatively rare)

- The algorithm supersedes the usual elimination scheme adopted for positive definite matrices (normal equations) based on Tinney II criterion, in which case 2×2 pivoting is not required. In fact the present implementation of the method is based on a code originally developed to deal with positive definite matrices.

- Whenever 2×2 pivoting has to be carried out, the selection of the companion row is critical to preserve the sparsity of the resulting triangular factors. The selection is made through an extended version of the Tinney II ordering scheme described next.

Ordering Criterion for 2×2 Pivoting

The motivation behind the Tinney II ordering scheme normally used for 1×1 pivoting is that when we pivot on the row with minimum degree (minimum number of nonzero elements) we are somehow minimizing the chances for the occurence of fill–ins. If, for example, the row being processed has n_k nonzero elements, including the diagonal, then $(n_k - 1)^2$ will be an upperbound for the number of possible fill–ins. Now, how can we generalize this idea to 2×2 pivotings? Though the minimum degree concept does not make much sense in this case, as simple examples will show, the related idea of minimizing the upperbound for the number of fill–ins can be extended to the 2×2 case.

Let us consider for example that rows k and i have been selected for the double pivoting; if we also consider that $d_k = 0$ and $d_i \neq 0$, the upperbound for the number of fill–ins is given by [17]

$$ub = (n_k + n_i - n_{ki} - 2)^2 - (n_i - n_{ki})^2 \qquad (20)$$

where n_k and n_i are the number of nonzero elements, including the diagonal elements, in rows k and i, respectively, and n_{ki} is the number of nonzero elements that appear in the same column in both rows (notice that eq. (20) does not include the fill–ins/cancelations that may occur in the double pivot row).

To give a flavor of how 2×2 pivoting affects sparsity as compared to the usual 1×1 pivoting let us consider the situation depicted in Fig. 3, in which we have $n_k = 5$, $n_i = 5$ and $n_{ki} = 4$. The upperbound for the number of fill–ins produced by the 2×2 pivoting of rows k and i is ub = 15. If d_k were not equal to zero, and pivoting on row k were performed using 1×1 pivot, then the upperbound would be ub = 16. Not too bad for the 2×2 pivot!

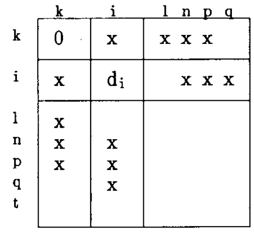

Fig.3: 2×2 pivoting with $n_{ki} = 4$

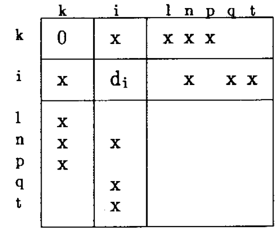

Fig.4: 2×2 pivoting with $n_{ki} = 3$

Now, if we had $n_{ki} = 3$ (rather than 4) as illustrates the example depicted in Fig.4, the upperbound would be ub = 21. This emphasizes the fact that in choosing the companion row one has to take into account not only the degree of the row but also the number of coincidences n_{ki}, as given by eq.(20).

Choosing the Companion Row

The detection of a null pivot during the factorization process does not necessarily mean a problem. The proper use of the 2×2 pivot, in additon to allowing the factorization to continue, may help in keeping the sparsity of the triangular factors: it is often the case that a 2×2 pivot, with one zero diagonal element, produces less fill–in elements than it would be produced by a sequence of two 1×1 pivotings, if that were possible. The key point here is the selection of the best companion row to a row with a zero 1×1 pivot (the 1×1 pivot is picked up by the standard Tinney II ordering criterion). In this section we give some further details about the selection of the companion row: particularly the main contribution of the paper regarding previous publications [12, 18] is emphasized.

As mentioned before, the 1×1 pivoting of the k–th row yields at most

$$ub = (n_k - 1)^2 \qquad (21)$$

fill–ins. The 2×2 pivoting of rows k and i may cause fill–ins not only in the remainder rows, but also in the two rows being pivoted (this is due to the normalization of rows k and i), as illustrated in Fig. 5. Both the upperbound, ub, and the number of fill–ins caused by normalization, nn, will depend on the diagonals of the 2×2 pivots:

i) $d_k \neq 0$, $d_i \neq 0$
$$ub = (n_k + n_i - n_{ki} - 2)^2 \qquad (22)$$
$$nn = n_k + n_i - 2n_{ki} - 2$$

ii) $d_k = 0$, $d_i \neq 0$
$$ub = (n_k + n_i - n_{ki} - 2)^2 - (n_i - n_{ki})^2 \qquad (23)$$
$$nn = n_k - n_{ki} - 2$$

iii) $d_k = d_i = 0$
$$ub = (n_k + n_i - n_{ki} - 2)^2 - (n_i - n_{ki})^2 - (n_k - n_{ki})^2 \qquad (24)$$
$$nn = -2$$

At each step of the factorization process, the Tinney II ordering scheme selects for pivoting the row with minimum ub which turns out to be the row with minimum degree. As for the 2×2 pivoting, whenever needed, the situation is a bit more complex, for choosing the pair of rows with minimum ub would require determining n_{ki} for a number of candidate pairs; even considering that 2×2 pivotings are relatively rare occurences, such an approach would be computationally prohibitive. So, simplifications are needed in order to keep sparsity without penalizing computational cost.

Ref. [18] suggests an extension to the Tinney II criterion by ignoring the coincidences between elements in the pivot rows; in terms of expressions (22)–(24) that is equivalent to assuming that $n_{ki}=2$, i.e. only the coincidences in the 2×2 pivot matrix are taken into account; all other possible coincidences are ignored. And so, the candidate pairs are ordered according to n_k+n_i-4, that is to say, according to the total number of elements in the two candidate rows, discounting the elements that form part of the 2×2 pivot matrix. Such an approximation may lead to very conservative results, as is shown in the following (see Illustrative Examples).

In this paper we take a different route: considering that the main motivation for using 2×2 pivots is the occurence of zero pivots when the standard 1×1 pivoting strategy plus Tinney II criterion are used, the proposed method takes advantage from the important fact that under these conditions at least one of the diagonal elements of the 2×2 pivot is zero. As is illustrated by Eqs. (22)–(24), the fact that either d_k or d_i is zero may significantly affect sparsity. The dynamic ordering scheme presented in the previous item of this section profits from this special characteristic of the problem being dealt with.

According to the proposed ordering scheme, 2×2 pivotings are carried out only in the event that the row selected for pivoting by Tinney II criterion presents zero, or very small, diagonal element. When that happens, rather than considering all pairs of rows as candidates for the 2×2 pivoting, the row selected by the standard Tinney II criterion (row k, which possibly presents a zero diagonal element) is considered one of the rows of the pair, and so the problem is limited to choosing a companion row (row i, according to the nomenclature adopted in the paper). As has been mentioned before, this is so because the existence of a zero diagonal in the 2×2 pivot is advantageous regarding the upperbound on the number of possible fill–in elements. Thus the method is able to profit from what otherwise would be a losing situation, i.e. the occurence of a zero pivot. The number of possible candidates can be further reduced if we consider some other characteristics of the state estimation problem, as is shown in the following.

Let's assume for a moment that the k–th row, which has zero diagonal ($d_k = 0$), has been selected for pivoting. Next we have to find the best companion row. The set of candidate companion rows is formed by the rows whose indices are the column indices of the elements in row k (i.e., in the matrix graph, the set of candidate companion rows is the set of nodes directly connected to node k); otherwise we would have a 2×2 pivot with zero determinant. Among the candidate rows it is selected the one which minimizes ub + nn, provided that the determinant of the 2×2 pivot is bigger than a given tolerance. The computational overhead is not significant for two main reasons: (1) row k is selected according to the minimum degree criterion, which guarantees that the size of the set of candidates is as small as possible, and (2) for realistic systems the need for 2×2 pivots is not very frequent (please refer to the test results section for typical figures). Notwithstanding that observation, the search for the companion row can be further speeded–up as discussed next.

In factorizing the gain matrix associated to Eq. (19), i.e. normal equations with equality constraints, most of the rows with zero diagonal are affected by previous pivotings in such a way that the diagonal elements become non–zero before its selection for pivoting. This is not guaranteed to happen in all cases, however. It may occur that one of these rows (row k) is selected for pivoting before having its diagonal modified by the processing of other rows. If so, usually the state equation (row l) associated to the bus with zero injection measurement is the best companion row. As a matter of fact, if we consider that there is no row j with $n_{kj}>n_{kl}$, than it can be proved that the upperbound associated to the pair (k,l) is the minimum one if there is no other candidate row j with $n_j<n_l$. Thus, only candidate rows with less elements than n_l need to be considered for pairing. This simple rule speeds up the search without sacrificing the objective of maximizing sparsity.

<u>Illustrative Examples</u>

The example matrices depicted in Fig. 5 illustrate how the sparsity is affected by: (1) the value (zero or non–zero) of the 2×2 pivot matrix diagonal elements; and (2) by the number of coincidences in the pivot rows.

For the sake of comparison, in all cases shown in Fig. 5 the number of elements in rows k and i remain the same: $n_k=5$ and $n_i=7$, respectively. What changes from case to case is either the number of coincidences or the value of the diagonal elements or both. Thus, in cases (a), (b), and (c), $n_{ki}=3$, which means that there is one coincident element in addition to the two elements that form part of the 2×2 pivot matrix; in cases (d), (e), and (f), the number of coincidences is five, i.e., $n_{ki}=5$; in cases (a) and (d) both diagonal elements are non–zero, i.e. $d_k \neq 0$ and $d_i \neq 0$; in cases (b) and (c), $d_k=0$ and $d_i \neq 0$; finally in cases (c) and (f) $d_k=d_i=0$.

$n_k=5$, $n_i=7$

```
k | d e e e     e     |        k | d e     e e e     |
i | e d     e e e e e |        i | e d     e e e e e |

       nki=3                          nki=5
```

$d_k \neq 0$, $d_i \neq 0$

```
k [ 1 0 n n i i n i i ]        k [ 1 0     n n n i i ]
i [ 0 1 i i n n n n n ]        i [ 0 1     n n n n n ]
  [ x   c c c c c c c ]          [                   ]
  [ x   c c c c c c c ]          [                   ]
  [   x c c c c c c c ]          [ x x     c c c c c ]
  [   x c c c c c c c ]          [ x x     c c c c c ]
  [ x x c c c c c c c ]          [ x x     c c c c c ]
  [   x c c c c c c c ]          [   x     c c c c c ]
  [   x c c c c c c c ]          [   x     c c c c c ]
   (a)  ub=49 nn=4                (d)  ub=25 nn=0
```

$d_k=0$, $d_i \neq 0$

```
k [ 1 0 n n i i n i i ]        k [ 1 0     n n n i i ]
i [ 0 1 i i 0 0 n 0 0 ]        i [ 0 1     n n n 0 0 ]
  [ x   c c c c c c c ]          [                   ]
  [ x   c c c c c c c ]          [                   ]
  [   x c c - - c - - ]          [ x x     c c c c c ]
  [   x c c - - c - - ]          [ x x     c c c c c ]
  [ x x c c c c c c c ]          [ x x     c c c c c ]
  [   x c c - - c - - ]          [   x     c c c - - ]
  [   x c c - - c - - ]          [   x     c c c - - ]
   (b)  ub=33 nn=0                (e)  ub=21 nn=-2
```

$d_k=0$, $d_i=0$

```
k [ 1 0 0 0 i i n i i ]        k [ 1 0     n n n i i ]
i [ 0 1 i i 0 0 n 0 0 ]        i [ 0 1     n n n 0 0 ]
  [ x   - - c c c c c ]          [                   ]
  [ x   - - c c c c c ]          [                   ]
  [   x c c - - c - - ]          [ x x     c c c c c ]
  [   x c c - - c - - ]          [ x x     c c c c c ]
  [ x x c c c c c c c ]          [ x x     c c c c c ]
  [   x c c - - c - - ]          [   x     c c c - - ]
  [   x c c - - c - - ]          [   x     c c c - - ]
   (c)  ub=29 nn=-2               (f)  ub=21 nn=-2
```

Fig.5 – Illustrative examples

where
- e element before pivoting
- x element zeroed by pivoting
- n normalized 'e' element
- i fill–in element in the pivot rows
- c contributions to the rest of the matrix
- – null contribution
- 0 eliminated element in the pivot rows

A dramatic variation in the upperbound ub is observed when we go from case (a) to case (f): sparsity improves with both number of coincidences and with the number of zero diagonal elements; roughly the same behavior is observed regarding the number, nn, of fill–in elements in the pivot rows.

It is noteworthy that the criterion suggested in Ref. [18] would fail in these cases: the upperbound would be ub=64 for cases (a) through (f); that is so because $ub=(n_k+n_i-4)^2$ for all the cases. On the other hand, the algorithm proposed in this paper clearly favors cases (e) and (f).

4. TEST RESULTS

Comparative studies have been carried out by testing on two real life systems.

System I: this is a 41 bus network with 58 branches and 14 zero injection buses; a typical measurement set for the network has 86 power flow measurements (86 pairs, P and Q power flows), 10 injection measurements (P and Q) and 10 voltage magnitude measurements (V); depending on the test case, the 14 zero injections may be treated either as pseudo–measurements or as equality constraints.

System II: this is a 1331 bus network with 1908 branches and 417 zero injection buses; a typical measurement system for this network has 3812 power flow measurements (pairs P,Q), 158 injection measurements (P, Q) and 158 voltage magnitude measurements; zero injections are treated as in the 41 bus system.

The first batch of test cases was aimed at comparing the performance of the proposed approach with the delayed elimination method. Table 1 is illustrative of the results obtained for System I with the measurement set described above and considering the zero injections as equality constraints, the gain matrix (eq. (19)) is an indefinite matrix of order 55 (fast decoupled state estimation matrices) with 294 off–diagonal elements. Table 1 gives the number of off–diagonal elements for both methods (delayed elimination and mixed pivoting). Though the delayed elimination scheme works fine in this example, the factorization process creates more than two times as much fill–ins as the mixed pivoting approach.

Method	Number of off–diagonal elements	
	Gain matrix (1/2)	Triangular factors
Delayed elimination	147	174
Mixed pivoting	147	159

Table 1: Sparsity of the triangular factors for System I

The second set of tests has been performed using System II and was aimed at illustrating as the sparsity of the triangular factors is affected by the presence of equality constraints. Table 2 compares two versions of the normal equations approach: zero injections treated as pseudomeasurements and as equality constraints. This table illustrates the fact that though the dimension of the sparse tableau formulations are bigger than the corresponding gain matrix with squared Jacobians for all types of measurements, the resulting triangular factors usually are sparser than the factors produced by the more compact formulations.

	case 1	case 2
Dimension of the gain matrix	1331	1748
Number of off–diagonal elements in the gain matrix (1/2)	3772	3531
Number of off–diagonal elements in the factors	5294	5233
Time to perform factorization (normalized)	1.09	1.00

Table 2: Sparsity of triangular factors for System II
Case 1: normal equations (eq. (6))
Case 2: equality constraints (eq. (17)) with mixed pivoting

The third batch of tests was designed to compare different approaches to the selection of the companion row for 2×2 pivoting. Table 4 is illustrative of the type of study that has been performed. Strategy A is the one described at the end of the previous section (the extended Tinney II scheme); Strategy B is a simplified version of Strategy A in which we force the equality constraint equation to be pivoted together with the corresponding state equation, provided that at the time either equation is chosen as candidate pivot row, the diagonal element of the equality constraint equation is still zero. Strategy C is a static version of Strategy B in which equality constraint equation and the corresponding state equation are always pivoted in pairs, regardless the present value of the diagonal elements (a fill–in may have occurred in the diagonal position of the equality constraint). Finally Strategy D simulates the method described in Ref. [12] to overcome nondefiniteness. Though in this reference there is no information about the companion row selection we assume the strategy used in Ref. [18]. Table 3 gives the numbers of elements of the triangular factors for the four strategies. Table 4 presents some computing time comparisons together with the number of 2×2 pivotings. All tests summarized in these tables have been performed with System II. Different cases correspond to different numbers and allocations of regular injection measurements; for example, notice that though case 4 has fewer injection measurements than case 5 it presents more off–diagonal elements, the reason being that in case 4 most of the injection measurements are located at buses adjacent to buses with zero injections represented as equality constraints.

		1	2	3	4	5	6
Number of injection measurements		158	258	187	229	359	458
No. off–diag. elements in H'H (1/2)		3531	4188	3691	4191	3796	3991
No. off–diag. elements in the factors	A	5233	6333	5502	6402	5647	5975
	B	5262	6395	5581	6554	5754	6127
	C	5248	6830	5621	7631	5978	6460
	D	5326	6517	5644	6488	5787	6061

Table 3: Effect of the selection of companion row (Strategies A, B, C and D) on the sparsity of the triangular factors for System II

		1	2	3	4	5	6
Factorization time (normalized)	A	1.00	1.00	1.00	1.00	1.00	1.00
	B	1.01	1.00	1.02	1.05	1.03	1.04
	C	0.97	1.09	1.05	1.41	1.06	1.14
	D	1.05	1.08	1.08	1.05	1.07	1.03
Number of 2×2 pivots	A	128	140	129	136	134	136
	B	128	140	130	138	136	138
	C	417	417	417	417	417	417
	D	129	140	130	137	136	137

Table 4: Factorization times and number of 2×2 pivotings using Strategies A, B, C and D for System II

5. CONCLUSIONS

The paper presents an efficient and robust method for dealing with indefinite matrices in power system state estimation. Indefinite matrices usually appear in connection with the normal equations method with equality constraints and with related tableau formulations such as the Hachtel's method. The method is based on the mixed 1×1, 2×2 pivoting strategy originaly suggested in [12]. The paper proposes a new and more efficient way to select the companion row in performing 2×2 pivots. Special designed routines that are able to profit from the specific features of the problem have been implemented and tested. All the routines have been implemented in such a way that when dealing with positive definite matrix the algorithm will behave like the standard Tinney II criterion. Even when applied to problems such as the state estimation with equality constraints, which leads to indefinite matrices, the proposed methods carries out factorization as if the matrices were positive definite, i.e. the standard factorization scheme is used: only when zero pivots are flagged, the method temporarily switches to 2×2 pivotings, which guarantees the progress of the factorization process as has been originally proved in [16]. Though the main body of the paper addresses the equality constrained normal equations method, the proposed technique is extendable to other tableau formulations.

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